

*The Cause of Encke's Division in Saturn's Ring.*

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§ 1. *Introductory.*

In a former paper,\* the effect of satellites in producing divisions in Saturn's Rings was discussed. The case taken was that where the satellite orbit and the rings were co-planar. The results afforded an explanation of the outer dimension of the Ring, Cassini's Division, the inner radius of the bright ring, and the existence of the Crêpe Ring. But no reason was given for the existence of Encke's Division, nor the numerous divisions reported by Lowell. In carrying the examination further, the fact that the satellite orbits are not precisely co-planar with the ring system must be considered. As given by the 'Annuaire du Bureau des Longitudes,' the inclination of the plane of the rings to the ecliptic is  $28^{\circ} 5'6''$ , while the corresponding inclinations of the satellite orbits are—

|                 |                       |             |                       |
|-----------------|-----------------------|-------------|-----------------------|
| Mimas .....     | $27^{\circ} 29'6''$ , | Dione ..... | $28^{\circ} 4'4''$ ,  |
| Enceladus ..... | $28^{\circ} 4'3''$ ,  | Rhea .....  | $28^{\circ} 22'8''$ , |
| Tethys .....    | $28^{\circ} 40'5''$ , | Titan ..... | $27^{\circ} 39'7''$ . |

In the cases of Mimas, Tethys, and Titan, the differences are distinctly marked, the first being  $36''$ , or  $0.01$  of a radian. The effect of this inclination is examined in the sequel.

The general hypotheses are the same as in the former paper, to which the reader is referred.

§ 2. *The Equations of Motion.*

Using the same notation as in the former paper, let  $M$  be the mass of the primary supposed situated at the origin of co-ordinates. It is surrounded by a single ring of  $p$  particles, whose unperturbed positions are in the  $xy$  plane, at the corners of a regular polygon. We shall take the co-ordinates of one of these particles,  $m_{\lambda}$ , as  $x_{\lambda}$ ,  $y_{\lambda}$ ,  $z_{\lambda}$ , or  $r_{\lambda}$ ,  $\theta_{\lambda}$ ,  $z_{\lambda}$ ;  $r$  and  $\theta$  being measured in the  $xy$  plane. The unperturbed values of these co-ordinates will be  $z_{\lambda}=0$ ,  $r_{\lambda}=\alpha$ ,  $\theta_{\lambda}=\omega t + \epsilon + \lambda 2\pi/p$ , where  $\omega^2 \alpha^3 = M + m_{\lambda} = M$ .

The perturbing satellite, of mass  $m'$ , describes a circular orbit of radius  $a'$  in a plane inclined at an angle  $\gamma$  to the  $xy$  plane. Any ellipticity of the primary  $M$  is neglected, so that the orbit of  $m'$  has stationary nodes. It

\* 'Phil. Trans.,' A, vol. 222, p. 101.

has been shown by Stroobant,\* that Saturn's Ring, as a whole, has little effect upon the movements of the satellites. We are therefore justified in treating the orbit of  $m'$  as an unperturbed circle. The co-ordinates of  $m'$  at time  $t$  will be

$$x' = a' \cos \theta', \quad y' = a' \sin \theta' \cos \gamma, \quad z' = a' \sin \theta' \sin \gamma,$$

where

$$\theta' = \omega' t + \epsilon',$$

and

$$\omega'^2 a'^3 = M + m' = M.$$

We have here taken the line of nodes as the  $x$ -axis, and have measured  $\theta'$  in the plane of the orbit.

The equations of motion are now, for the particle  $m_\lambda$

$$\left. \begin{aligned} \ddot{r}_\lambda - r_\lambda \dot{\theta}_\lambda^2 &= \frac{\partial F}{\partial r_\lambda}, \\ \frac{1}{r_\lambda} \frac{d}{dt} (r_\lambda^2 \dot{\theta}_\lambda) &= \frac{1}{r_\lambda} \frac{\partial F}{\partial \theta_\lambda}, \\ \ddot{z} &= \frac{\partial F}{\partial z_\lambda}. \end{aligned} \right\} \quad (1)$$

In these

$$F \equiv \frac{M + m_\lambda}{(r_\lambda^2 + z_\lambda^2)^{1/2}} + \frac{m'}{\Delta_\lambda} - \frac{m'}{a'^3} (x_\lambda x' + y_\lambda y' + z_\lambda z') \\ + \sum_\mu \frac{m_\mu}{D_{\lambda\mu}} - \sum_\mu \frac{m_\mu (x_\lambda x_\mu + y_\lambda y_\mu + z_\lambda z_\mu)}{(r_\mu^2 + z_\mu^2)^{3/2}},$$

where

$$\Delta_\lambda^2 = (x_\lambda - x')^2 + (y_\lambda - y')^2 + (z_\lambda - z')^2,$$

and

$$D_{\lambda\mu}^2 = (x_\lambda - x_\mu)^2 + (y_\lambda - y_\mu)^2 + (z_\lambda - z_\mu)^2 \\ = r_\lambda^2 + r_\mu^2 - 2r_\lambda r_\mu \cos(\theta_\lambda - \theta_\mu) + (z_\lambda - z_\mu)^2.$$

In the case of Saturn's Rings, the particles forming any ring can only perform small vibrations in the plane of the ring about their zero positions if collision with other particles is to be avoided. We therefore assume

$$r_\lambda = a(1 + \rho_\lambda),$$

and

$$\theta_\lambda = \omega t + \epsilon + \lambda 2\pi/p + \sigma_\lambda,$$

where  $\rho_\lambda$  and  $\sigma_\lambda$  are small.

When the orbit of the perturbing satellite is in the  $xy$  plane,  $z_\lambda = 0$  is a solution of the equations (1). Further, as we shall only consider the case where the inclination of the satellite orbit is small (and therefore we may replace  $\sin \gamma$  by  $\gamma$  and  $\cos \gamma$  by unity), we assume

$$z_\lambda = a\gamma\zeta_\lambda,$$

$\zeta_\lambda$  being also small.

\* 'Comptes Rendus,' vol. 172, p. 913.

The equations (1) now become

$$\begin{aligned}
 a\ddot{\rho}_\lambda - 2a\omega\dot{\sigma}_\lambda - \omega^2\rho_\lambda - a\omega^2 &= \left(\frac{\partial F}{\partial r_\lambda}\right)_0 + \sum_\mu a\rho_\mu \left(\frac{\partial^2 F}{\partial r_\lambda \partial r_\mu}\right)_0 \\
 &\quad + \sum_\mu \sigma_\mu \left(\frac{\partial^2 F}{\partial r_\lambda \partial \theta_\mu}\right)_0 + \sum_\mu a\gamma\zeta_\mu \left(\frac{\partial^2 F}{\partial r_\lambda \partial z_\mu}\right)_0, \\
 a\ddot{\sigma}_\lambda + 2\omega a\dot{\rho}_\lambda &= \left(\frac{\partial F}{\partial r_\lambda \partial \theta_\lambda}\right)_0 + \sum_\mu a\rho_\mu \left(\frac{\partial^2 F}{\partial r_\mu \cdot r_\lambda \partial \theta_\lambda}\right)_0 \\
 &\quad + \sum_\mu \sigma_\mu \left(\frac{\partial^2 F}{\partial \theta_\mu \cdot r_\lambda \partial \theta_\lambda}\right)_0 + \sum_\mu a\gamma\zeta_\mu \left(\frac{\partial^2 F}{\partial z_\mu \cdot r_\lambda \partial \theta_\lambda}\right)_0, \\
 a\gamma\ddot{\zeta}_\lambda &= \left(\frac{\partial F}{\partial z_\lambda}\right)_0 + \sum_\mu a\rho_\mu \left(\frac{\partial^2 F}{\partial r_\mu \partial z_\lambda}\right)_0 \\
 &\quad + \sum_\mu \sigma_\mu \left(\frac{\partial^2 F}{\partial \theta_\mu \partial z_\lambda}\right)_0 + \sum_\mu a\gamma\zeta_\mu \left(\frac{\partial^2 F}{\partial z_\mu \partial z_\lambda}\right)_0. \quad (2)
 \end{aligned}$$

Now

$$\Delta_\lambda^2 = r_\lambda^2 + z_\lambda^2 + a'^2 - 2r_\lambda a' \{\cos \theta_\lambda \cos \theta' + \sin \theta_\lambda \sin \theta' \cos \gamma\} - 2z_\lambda z'.$$

Since  $\cos \gamma$  is replaced by unity, and in the formation of zero values of the derivatives we put  $z_\lambda = 0$ , it is clear that, wherever it appears in the equations (2),  $\Delta_\lambda$  has the same value as in the equations of the former paper.

In the same way the other zero values are

$$D_{\lambda\mu}^2 = 4a^2 \sin^2(\lambda - \mu) \pi/p,$$

$$x_\lambda x' + y_\lambda y' + z_\lambda z' = aa' \cos(\omega't + \epsilon' - \omega t - \epsilon - \lambda 2\pi/p),$$

and

$$x_\lambda x_\mu + y_\lambda y_\mu + z_\lambda z_\mu = a^2 \cos(\lambda - \mu) 2\pi/p.$$

It is evident, then, that the terms of equations (2) will all be the same as those of the former paper except where  $z$ -derivatives appear. The latter terms are now evaluated:—

$$\begin{aligned}
 \sum_\mu a\gamma\zeta_\mu \left(\frac{\partial^2 F}{\partial z_\mu \partial r_\lambda}\right)_0 &= -\frac{3m'aa'}{\Delta_\lambda^5} \gamma^2 \{a - a' \cos(\theta' - \theta_\lambda)\} \sin \theta' \zeta_\lambda \\
 &= 0, \text{ to the order of value we are using.} \\
 \sum_\mu a\gamma\zeta_\mu \left(\frac{\partial^2 F}{\partial z_\mu \cdot r_\lambda \partial \theta_\lambda}\right)_0 &= \frac{3m'aa'^2\gamma^2 \sin \theta'}{\Delta_\lambda^5} \sin(\theta' - \theta_\lambda) \zeta_\lambda \\
 &= 0, \\
 \left(\frac{\partial F}{\partial z_\lambda}\right)_0 &= m' \sin \theta' \cdot \gamma \left\{ \frac{a'}{\Delta_\lambda^3} - \frac{1}{a'^2} + \frac{3a}{a'^3} \cos(\theta' - \theta_\lambda) \right\}, \\
 \sum_\mu a\rho_\mu \left(\frac{\partial^2 F}{\partial z_\lambda \partial r_\mu}\right)_0 &= 3m' a\gamma\rho_\lambda \sin \theta' \left\{ -\frac{aa' - a'^2 \cos(\theta' - \theta_\lambda)}{\Delta_\lambda^5} + \frac{a}{a'^4} \cos(\theta' - \theta_\lambda) \right\}, \\
 \sum_\mu \sigma_\mu \left(\frac{\partial^2 F}{\partial \theta_\mu \partial z_\lambda}\right)_0 &= 3m' a\gamma\sigma_\lambda \sin \theta' \left\{ \frac{a'^2}{\Delta_\lambda^5} \sin(\theta' - \theta_\lambda) + \frac{1}{a'^3} \sin(\theta' - \theta_\lambda) \right\}, \\
 \sum_\mu a\gamma\zeta_\mu \left(\frac{\partial^2 F}{\partial z_\mu \partial z_\lambda}\right)_0 &= \left[ -\frac{M + m_\lambda}{a^3} - \frac{m'}{\Delta_\lambda^3} - \sum_\mu \frac{m_\mu}{D_{\lambda\mu}^3} \right] a\gamma\zeta_\lambda + \sum_\mu \frac{m_\mu}{D_{\lambda\mu}^3} \zeta_\mu.
 \end{aligned}$$

In the last expression the  $\mu$ -summation in the right-hand member does not include the case  $\mu = \lambda$ . Also  $\theta'$  now equals  $\omega't + \epsilon'$  and  $\theta_\lambda$  equals  $\omega t + \epsilon + \lambda 2\pi/p$ .

It is clear from these results that the first pair of equations (2) remain unchanged by the small inclination of the satellite orbit. The effect of this inclination is only to produce vibrations normal to the plane of the ring.

We are therefore left with the single set of equations

$$\begin{aligned} \ddot{\xi}_\lambda = m' \sin \theta' & \left\{ \frac{a'}{a\Delta_\lambda^3} - \frac{1}{aa'^2} + \frac{3}{a'^3} \cos(\theta' - \theta_\lambda) \right\} \\ & + 3m' \sin \theta' \left\{ -\frac{aa' - a'^2 \cos(\theta' - \theta_\lambda)}{\Delta_\lambda^5} + \frac{a}{a'^4} \cos(\theta' - \theta_\lambda) \right\} \rho_\lambda \\ & + 3m' \sin \theta' \left\{ \frac{a'^2}{\Delta_\lambda^5} \sin(\theta' - \theta_\lambda) + \frac{1}{a'^3} \sin(\theta' - \theta_\lambda) \right\} \sigma_\lambda \\ & + \left\{ -\frac{M + m_\lambda}{a^3} - \frac{m'}{\Delta_\lambda^3} - \sum_\mu \frac{m_\mu}{D_{\lambda\mu}^3} \right\} \zeta_\lambda + \sum_\mu \frac{m_\mu}{D_{\lambda\mu}^3} \zeta_\mu. \end{aligned} \quad (3)$$

It is to be noted that this equation holds for all integral values of  $\lambda$  from 1 to  $p$ . It is independent of  $\gamma$ ; and the quantities  $\rho_\lambda, \sigma_\lambda$  are known for all values of  $\lambda$ , having been determined from the first two of equations (1) which have been solved to the necessary degree of approximation previously.

In the former paper, certain approximations were made to render possible the solution of the period equation without a definite knowledge of the separate masses forming a ring.\* This approximation amounted to the use of the mean position instead of the actual position of a particle in calculating its perturbing effect upon another particle. If we adopt the same process again, we reject the terms  $\sum_\mu \frac{m_\mu}{D_{\lambda\mu}^3} \zeta_\mu$  in equation (3). We have then a series of  $p$  equations for the motion of the  $p$  particles, each equation containing the co-ordinates of the particle under consideration and not the co-ordinates of the other particles. This of course very much simplifies the work.

We may write

$$\begin{aligned} \Delta_\lambda^{-3} &= a'^{-3} \sum_i c_i \cos i(\theta' - \theta_\lambda), \\ \Delta_\lambda^{-5} &= a'^{-5} \sum_i d_i \cos i(\theta' - \theta_\lambda). \end{aligned}$$

On substituting these and reducing the expressions, we find that equations (3) may be written

$$\begin{aligned} \ddot{\xi}_\lambda + \zeta_\lambda \sum_j \Phi_{1,j} \cos j(\theta' - \theta_\lambda) + \rho_\lambda \sum_j \Phi_{2,j} \sin \{(j+1)\theta' - j\theta_\lambda\} \\ + \sigma_\lambda \sum_j \Phi_{3,j} \cos \{(j+1)\theta' - j\theta_\lambda\} = \sum_j \Phi_{4,j} \sin \{(j+1)\theta' - j\theta_\lambda\}. \end{aligned} \quad (4)$$

\* *Loc. cit.*, pp. 123 and 124.

In these equations the values of the coefficients  $\Phi$  are readily obtained. In particular

$$\Phi_{1,0} = \omega^2 + \omega^2 \sum_{\mu} \frac{m_{\mu}}{M} \frac{\frac{1}{8}}{\sin^3(\lambda - \mu) \pi/p} + \nu' \omega'^2 c_0.$$

As was shown in the previous paper,\*  $\sum_{\mu} \frac{m_{\mu}}{M} \frac{1}{8 \sin^3(\lambda - \mu) \pi/p}$  ranges in value from zero to  $0.0096 p^3 \bar{m}/M$ , where  $\bar{m}$  is the maximum mass of a particle appearing in the ring. All the quantities  $\Phi$  except  $\Phi_{1,0}$  are factored by  $\nu'$  or  $m'/M$ . As this quantity is very small, all the quantities  $\Phi$  except  $\Phi_{1,0}$  will be small, and their squares and products may be neglected. Also in  $\Phi_{1,0}$  the term  $\nu' \omega'^2 c_0$  may be rejected in comparison with the other two. We shall then write

$$\Phi_{1,0} = \omega^2 + \omega^2 F_{\lambda}.$$

### § 3. *The Solution of the Equations.*

In solving equations (4) let us first take the complementary function. The equation to be solved is

$$\ddot{\xi} + \zeta [\sum_j \Phi_{1,j} \cos j(\omega't + \epsilon' - \omega t - \epsilon - \lambda 2\pi/p)] = 0. \quad (5)$$

The solutions of the system are obtained by giving to  $\lambda$  its  $p$  values 1, 2, ...  $p$ . The equation is of Hill's well-known form and can readily be solved by the method of Whittaker and Ince.† To simplify the form, change the independent variable from  $t$  to  $\phi$  which equals  $\omega't + \epsilon' - \omega t - \epsilon - \lambda 2\pi/p$ . Then

$$\zeta'' + \zeta \left[ \frac{1}{(\omega' - \omega)^2} \sum_j \Phi_{1,j} \cos j\phi \right] = 0. \quad (6)$$

Let  $\zeta = e^{c\phi} u$ , where  $u$  is a periodic function of the independent variable  $\phi$ . Then

$$c^2 u + 2cu' + u'' + u \left[ \frac{1}{(\omega' - \omega)^2} \sum_j \Phi_{1,j} \cos j\phi \right] = 0. \quad (7)$$

Also let

$$u = \cos(n\phi - \tau) + \sum_i P_i \Phi_{1,i} \dots,$$

$$c = \sum_i Q_i \Phi_{1,i} \dots,$$

$$\Phi_{1,0} = n^2(\omega' - \omega)^2 + \sum_i R_i \Phi_{1,i} \dots$$

In the summations, all integral values of  $i$  are to be taken except  $i = 0$ . The general case should include products of the quantities  $\Phi$ , but as has been pointed out these may be neglected. The quantities  $Q_i$  and  $R_i$  are constants,

\* P. 123.

† 'Monthly Notices, R.A.S.,' LXXV, 5, p. 436.

and  $P_i$  is a periodic function of  $\phi$ . If we add the two conditions: (1) That no term  $\sin(n\phi - \tau)$  may appear in the expression for  $u$ , and (2) that  $u$  is purely periodic, we are able to determine completely all the coefficients entering into the expressions.

On substituting in equations (7) and equating to zero the coefficients of those terms which are independent of  $\Phi$ , we, have an identity. From the coefficient of  $\Phi_{1,i}$  we find

$$-2n Q_i \sin(n\phi - \tau) + P_i'' + n^2 P_i + \frac{R_i}{(\omega' - \omega)^2} \cos(n\phi - \tau) + \frac{\cos(n\phi - \tau) \cos i\phi}{(\omega' - \omega)^2} = 0. \quad (8)$$

In general, therefore,

$$Q_i = 0, \quad R_i = 0, \\ P_i = \frac{\frac{1}{2} \cos \{(i+n)\phi - \tau\}}{(\omega' - \omega)^2 \{(i+n)^2 - n^2\}} + \frac{\frac{1}{2} \cos \{(i-n)\phi - \tau\}}{(\omega' - \omega)^2 \{(i-n)^2 - n^2\}}. \quad (9)$$

But in the particular case of  $i = 2n$ , equation (8) reduces to

$$-2n Q_{2n} \sin(n\phi - \tau) + P_{2n}'' + n^2 P_{2n} + \frac{R_{2n}}{(\omega' - \omega)^2} \cos(n\phi - \tau) + \frac{1}{(\omega' - \omega)^2} \left\{ \frac{1}{2} \cos(3n\phi - \tau) + \frac{1}{2} \cos(n\phi - \tau) \cos 2\tau - \frac{1}{2} \sin(n\phi - \tau) \sin 2\tau \right\} = 0.$$

Whence, to fulfil the given conditions,

$$Q_{2n} = -\frac{\sin 2\tau}{4n(\omega' - \omega)^2}, \\ R_{2n} = -\frac{1}{2} \cos 2\tau, \\ P_{2n} = \frac{\cos(3n\phi - \tau)}{16n^2(\omega' - \omega)^2}. \quad (10)$$

We have, then,

$$c = -\frac{\sin 2\tau}{4n(\omega' - \omega)^2} \Phi_{1,2n},$$

and  $\Phi_{1,0} = n^2(\omega' - \omega)^2 - \frac{1}{2} \Phi_{1,2n} \cos 2\tau. \quad (11)$

These two expressions, together with (9) and (10), constitute a solution valid in the vicinity of  $\Phi_{1,0} = n^2(\omega' - \omega)^2$ , where  $n$  is any integer. The parameter  $\tau$  is determined from the second of equations (11). The value so obtained, substituted in the expression for  $c$ , determines that quantity completely. It is clear that real values of  $\tau$  mean real values of  $c$ , and hence vibrations of increasing amplitude or instability. Real values of  $\tau$  can only occur when

$$|\Phi_{1,0} - n^2(\omega' - \omega)^2| \leq \left| \frac{1}{2} \Phi_{1,2n} \right|;$$

and owing to the extreme smallness of  $\Phi_{1,2n}$ , this is, with a close degree of approximation, when

$$\Phi_{1,0} - n^2(\omega' - \omega)^2 \equiv 0. \quad (12)$$

We pass next to the particular integral of equation (4). The values of  $\rho_\lambda$  and  $\sigma_\lambda$  to be used are obtained from the former paper. From p. 110, we have

$$\rho = e^{c\phi} \{A_0 \sin(n\phi - \tau) + \Sigma \Sigma A_{r,s} \Theta_{r,s} \dots\},$$

and

$$\sigma = e^{c\phi} \{X_0 \cos(n\phi - \tau) + \Sigma \Sigma X_{r,s} \Theta_{r,s} \dots\}.$$

In these expressions  $A_0$  and  $X_0$  are related constants, and  $A_{r,s}$ ,  $X_{r,s}$  are determined later in the paper. As, however  $\Theta_{r,s}$  is of the same order as  $\Phi_{2,j}$  and  $\Phi_{3,j}$ , it is evident that on substituting in equation (4) only the first terms of the expressions for  $\rho$  and  $\sigma$  need be used. We have further to add the particular integrals for  $\rho$  and  $\sigma$  given on p. 117 of the former paper. As again, these particular integrals are of the same order of value as the quantities  $\Phi_{2,j}$  and  $\Phi_{3,j}$  it is clear that we may omit them in making the substitutions. The final result is that the values of  $\rho$  and  $\sigma$  will both produce terms of the type

$$\Omega_i \cos \{(i+1)\theta' - i\theta_\lambda\} \cdot \exp. c(\theta' - \theta_\lambda),$$

where  $i$  may take all integral values, positive and negative, and  $c$  and  $\Omega_i$  are known constants of the same order as  $\Phi_{2,j}$  and  $\Phi_{3,j}$ .

Particular integrals will arise from these terms and also from the remaining terms

$$\Phi_{4,j} \sin \{(j+1)\theta' - j\theta_\lambda\}.$$

Taking the first of these we have to find the particular integral of

$$\xi_\lambda + \xi_\lambda \sum_j \Phi_{1,j} \cos j(\theta' - \theta_\lambda) = \Omega_i \cos \{(i+1)\theta' - i\theta_\lambda\} \exp. c(\theta' - \theta_\lambda). \quad (13)$$

Removing  $\exp. \{c(\theta' - \theta_\lambda)\}$ , and neglecting  $c^2$ , we have

$$2c(\omega' - \omega) \xi_\lambda + \xi_\lambda + \xi_\lambda \sum_j \Phi_{1,j} \cos j(\theta' - \theta_\lambda) = \Omega_i \cos \{(i+1)\theta' - i\theta_\lambda\}.$$

On working out the particular integral in the way previously adopted,\* and noting that products of  $\Phi$  and  $\Omega$  may be rejected, we have

$$\xi_\lambda = \frac{\exp. \{c(\theta' - \theta_\lambda)\} \Omega_i}{\Phi_{1,0} - \{(i+1)\omega' - i\omega\}^2} \left\{ \cos \{(i+1)\theta' - i\theta_\lambda\} + \frac{2c(\omega' - \omega) \{(i+1)\omega' - i\omega\}}{\Phi_{1,0} - \{(i+1)\omega' - i\omega\}^2} \sin \{(i+1)\theta' - i\theta_\lambda\} \right\},$$

or, to the same order of values, rejecting the product  $c\Omega_i$ ,

$$\xi_\lambda = \frac{\exp. \{c(\theta' - \theta_\lambda)\} \Omega_i}{\Phi_{1,0} - \{(i+1)\omega' - i\omega\}^2} \cos \{(i+1)\theta' - i\theta_\lambda\}. \quad (14)$$

\* *Loc. cit.*, p. 116.

To this must be added the particular integral of

$$\xi_{\lambda} + \xi_{\lambda} \sum_j \Phi_{1,j} \cos j(\theta' - \theta_{\lambda}) = \Phi_{4,j} \sin \{(j+1)\theta' - j\theta_{\lambda}\}. \quad (15)$$

To the same order of approximation, the required integral is

$$\xi_{\lambda} = \frac{\Phi_{4,j} \sin \{(j+1)\theta' - j\theta_{\lambda}\}}{\Phi_{1,0} - \{(j+1)\omega' - j\omega\}^2}. \quad (16)$$

Finally, the whole particular integral is represented by the sum of (14) and (16).

#### § 4. *Discussion of the Results.*

Expression (12), with the meaning of  $\Phi_{1,0}$  inserted, is

$$\omega^2 + \omega^2 F_{\lambda} = n^2 (\omega' - \omega)^2. \quad (17)$$

As already pointed out, for the differing values of  $\lambda$ ,  $F_{\lambda}$  ranges from zero to  $0.0096 p^3 \bar{m}/M$ . As in the previous paper, we shall treat  $F_{\lambda}$  as a parameter varying continuously between the limits mentioned. Solving (17) and taking only the case where  $\omega > \omega'$ ,

$$\frac{\omega'}{\omega} = \frac{n - \sqrt{(1 + F_{\lambda})}}{n}. \quad (18)$$

The maximum value of  $F_{\lambda}$  as determined from the observed dimensions of Cassini's Division is 0.0173. The following Table gives the positions of the divisions resulting from the instability :—

|         | $F_{\lambda} = 0.$ |         | $F_{\lambda} = 0.0173.$ |         |
|---------|--------------------|---------|-------------------------|---------|
|         | $\omega'/\omega.$  | $a/a'.$ | $\omega'/\omega.$       | $a/a'.$ |
| $n = 1$ | 0                  | 0       | —                       | —       |
| 2       | $\frac{1}{2}$      | 0.630   | 0.4958                  | 0.626   |
| 3       | $\frac{2}{3}$      | 0.763   | 0.6639                  | 0.761   |

Applying these results to the case of satellite Mimas, we should find, from the solution  $n = 2$ , a narrow division in Saturn's Ring extending from 16.9'' to 16.8''. This is just at the inner edge of Cassini's Division. For Enceladus the same solution would give a narrow division from 21.7'' to 21.6''. This is outside the limits of the existing ring. The solution for  $n = 3$  gives for Mimas a narrow division at 20.5''. This is again just outside the existing ring.

The normal vibrations represented by the solution (17) do not lead, then, to any new divisions in the Ring of Saturn. The particular integral, represented by the sum of (14) and (16), becomes large, and instability ensues in the vicinity of that position where

$$\Phi_{1,0} - \{(i+1)\omega' - i\omega\}^2 = 0. \quad (19)$$

Taking the value of  $\Phi_{1,0}$ , equation (19) becomes

$$\omega^2(1+F_\lambda) - \{(i+1)\omega' - i\omega\}^2 = 0.$$

The solution of this equation for which  $\omega > \omega'$  is

$$\frac{\omega'}{\omega} = \frac{i - \sqrt{(1+F_\lambda)}}{i+1}. \quad (20)$$

As before,  $F_\lambda$  ranges in value from 0 to 0.0173. The following Table gives the limits of the zones of instability arising from (20) for various values of  $i$ , which is an integer :—

|         | $F_\lambda = 0.$  |         | $F_\lambda = 0.0173.$ |         |
|---------|-------------------|---------|-----------------------|---------|
|         | $\omega'/\omega.$ | $a/a'.$ | $\omega'/\omega.$     | $a/a'.$ |
| $i = 1$ | 0                 | 0       | —                     | —       |
| 2       | $\frac{1}{3}$     | 0.481   | 0.331                 | 0.478   |
| 3       | $\frac{1}{2}$     | 0.630   | 0.498                 | 0.628   |
| 4       | $\frac{2}{3}$     | 0.711   | 0.598                 | 0.710   |
| 5       | $\frac{3}{4}$     | 0.763   | 0.665                 | 0.762   |

As appears in this Table, a series of narrow divisions arise from this part of the integral.

On applying the ratios to the satellite Mimas, we have the following series of divisions. They are worked to two places of decimals, in order to show the order of magnitude of the width of the divisions.

For  $n = 2$ , 12.90'' to 12.81'',

$n = 3$ , 16.89'' to 16.85'',

$n = 4$ , 19.08'' to 19.03'',

$n = 5$ , 20.50'' to 20.45''.

The first division is in the Crêpe Ring, just within the commencement of Ring B, the inner diameter of which is 13.00''.

The second division is at the inner edge of Cassini's Division.

The third division is a new one and falls almost exactly in the position given for Encke's Division. Lowell\* gives the radius of Encke's Division as 19.00''; while See† states it at 18.87'' with a width of 0.41''. In either case theory clearly indicates the existence of this division.

The third division again falls outside the existing ring.

Applying next the ratios in the Table to the satellite Enceladus, the only case to be mentioned is the value when  $n = 2$ . This should give a division extending from 16.56'' to 16.44''.

\* 'Lowell Observatory Bulletin,' No. 68.

† 'Astr. Nach.,' No. 3768.

There is no observed division at this position that I have been able to find. But it is easy to show that it must be very indefinite. The value of the inclination of the orbit of Enceladus is  $1\cdot3'$  compared with  $36'$  for that of Mimas. The effect of the inclination must, therefore, be less marked in the former case. Again, if we compare the distance of the satellite producing the Encke Division from that division with that of Enceladus from the new division, taking ratios (as that is the form in which the distances appear in the analysis) we find  $a/a' = 0\cdot7$  for Encke's Division and  $a/a' = 0\cdot48$  for the new division. The perturbing effect of Mimas in Encke's Division is, therefore, much greater than that of Enceladus in the new division. On these two counts we should expect the new division to be less distinct than Encke's Division. But Encke's Division is only visible with difficulty; hence it is to be expected that the new division should be unobservable.

None of the remaining satellites produce divisions corresponding to the formulæ of this paper within the existing Ring system.

#### § 5. *Summary.*

The effect of a satellite moving in an orbit inclined to the plane of the Rings of Saturn is to produce one new division. If the satellite be taken as Mimas, a narrow division will be caused at a distance  $19\cdot05''$  from Saturn. This closely corresponds with the observations of Encke's Division.

Enceladus should produce a division at  $16\cdot5''$ . No observed division is recorded at this place, but because of the minute inclination of the orbit of Enceladus and because of this satellite's distance from this place, it is to be expected that the division will be comparatively feeble.

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